ELECTRICAL NETWORKS AND STEPHENSON'S CONJECTURE

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We dedicate this article to the memory of Bill Thurston.

Abstract. In this paper, we consider a planar annulus, i.e., a bounded, 2-connected, Jordan domain, endowed with a sequence of triangulations exhausting it. We then construct a corresponding sequence of maps which converge uniformly on compact subsets of the domain, to a conformal homeomorphism onto the interior of a Euclidean annulus bounded by two concentric circles. As an application, we will affirm a conjecture raised by Ken Stephenson in the 90's which predicts that the Riemann mapping can be approximated by a sequence of electrical networks.

0. Introduction

0.1. **Riemann's Mapping Theorem and Thurston's disk packing scheme.** The Riemann Mapping Theorem asserts that any simply connected planar domain which is not the whole plane, can be mapped homeomorphically by a conformal mapping onto the open unit disk. That is, the domains are *conformally equivalent*. After a suitable normalization, this mapping is called the Riemann mapping and it is desirable to have a concrete approximation of it. In [48], Rodin and Sullivan proved Thurston's celebrated conjecture [56] asserting that a scheme based on the Koebe-Andreev-Thurston disk packing theorem (cf. [1, 2, 43, 57]) converges to the Riemann mapping.

In order to formulate Thurston's conjecture which inspired Stephenson's conjecture, we need to recall a few definitions. Let P be a *disk packing* in the complex plane $\mathbb{C}$. An *interstice* is a connected component of the complement of P; and one whose closure intersects only three disks in P is called a *triangular interstice*. We will let $\operatorname{supp}(P)$ denote the union of the disks in P and all its bounded interstices. The disks of P that intersect the boundary of its support are called *boundary disks*. Two finite disk packings P and $\tilde{P}$ in $\mathbb{C}$ will be called *isomorphic* if there exists an orientation preserving homeomorphism $\phi : \operatorname{supp}(P) \to \operatorname{supp}(\tilde{P})$ such that $\phi(P) = \tilde{P}$. It is evident that such an isomorphism h induces a bijection between the disks of P and the disks of $\tilde{P}$ and that an isomorphism of packing is a combinatorial notion.

Let $\Omega \subsetneq \mathbb{C}$ be a bounded simply connected domain, and let p_0 be an interior point in it. For each positive integer n, let P^n be a disk packing in Ω in which all bounded interstices are triangular. Assume that there is a sequence of positive numbers δ_n which converges to zero, such that: i) the radius of every disk in P^n is smaller than δ_n, and ii) every boundary

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disk in P^n is at distance at most δ_n from $\partial\Omega$. Finally, let P_0^n be a selected disk in P^n which is closest to p_0 or contains it.

The Disk Packing Theorem (Koebe-Andreev-Thurston) implies that there exists an isomorphic packing $\tilde{P}^n$ in the closed unit disk $\bar{\mathbb{D}}$ with all of its boundary disks tangent to the unit circle $\mathbb{S}^1$. Furthermore, if the given graph is isomorphic to the 1-skeleton of a triangulation of the Riemann sphere, then the packing is unique up to applying a Möbius transformation. Let

$$f_n : \mathrm{supp}(P^n) \to \mathrm{supp}(\tilde{P}^n) \tag{0.1}$$

be an isomorphism of P^n and $\tilde{P}^n$. Furthermore, normalize $\tilde{P}^n$ by a sequence of Möbius transformations preserving U so that $\tilde{P}_0^n$, the disk corresponding to P_0^n, is centered at the origin. Thurston conjectured that if the packings P^n are chosen to be sub packings of scaled copies of the infinite hexagonal disk packing of $\mathbb{C}$, then the sequence of piecewise affine maps (i.e., simplicial) f_n converges uniformly on compact subsets of Ω to the Riemann mapping from Ω to $\mathbb{D}$.

Rodin and Sullivan [48] proved Thurston's Conjecture by first showing that the maps f_n are K-quasiconformal, for some fixed K. Hence, there exists a subsequence which will converge to a limit function f which must also be K-quasiconformal. Rodin and Sullivan further showed that f must be 1-quasiconformal, and therefore, f is in fact conformal. He and Schramm [33, Theorem 1.1] developed profound techniques which avoid the machinery of quasiconformal mapping that is heavily used in Rodin-Sullivan's proof. Up to date, their theorem and advances [34] in the simply connected case, is the most advanced (see also their related work in [35]).

More recently, Chow and Luo [18] found applications of circle mappings to the study of discrete Ricci flow on surfaces; see also the work of Glickenstein [27] for related study. There are also applications of circle packings to algorithmic computer vision and computational conformal geometry due to Gu, Luo and Yau, Gu, Zeng, Luo and Yau, and Sass, Stephenson and Brock (cf. [31, 32] and [49] as examples and further advances).

0.2. **Electrical networks and Stephenson's conjecture.** In his attempts to prove *uniformization*, Riemann suggested considering a planar annulus as made of a uniform conducting metal plate. When one applies voltage to the plate, keeping one boundary component at a fixed voltage k and the other at the voltage 0, *electrical current* will flow through the annulus. The *equipotential* sets form a family of disjoint simple closed curves foliating the annulus and separating the boundary curves. The *current* flow sets consist of simple disjoint arcs connecting the boundary components, and they as well foliate the annulus. Together, the two families provide curved "rectangular" coordinates on the annulus that can be used to turn it into a right circular cylinder, or into a (conformally equivalent) circular concentric annulus.

An *electrical circuit* or *network* is a collection of nodes and connecting wires. For instance, a disk packing of a fixed planar domain induces such a network where each center of a disk corresponds to a node and a wire connects each pair of nodes corresponding to tangent boundaries. It is therefore reasonable to conjecture that if the domain is made of thin conducting material then its electrical behavior can be approximated by a sequence of networks that approximates its *shape*.

Stephenson's conjecture from the 90's is concerned with constructing such an approximation:

Conjecture 0.2 (Stephenson [54]). *Given a sequence of networks approximating a simply-connected, bounded, Jordan domain arising, for instance, from a sequence of disk packing, choose conductance constants along the edges (for each network) according to Equation (1.2). Then the sequence of discrete potentials and currents will converge to the ones induced by the Riemann mapping.*

We have phrased this conjecture in the more recent formulation of (1.5) (see Section 1 for the details). In fact, a similar conjecture can as well be formulated for any domain that can be approximated (in a sense that we will define in Section 3.1) by a sequence of (refined) quasi-uniform triangulations (see Definition A.7).

In Theorem 3.12, we will formalize and affirm Stephenson's conjecture in the case of a polygonal annulus by methods that are different from the ones used in his paper or those mentioned in Section 0.1. In particular, we will explore a large class of networks for which it holds. As one application, we will affirm Stephenson's conjecture in its original form.

0.3. **The themes of this paper.** There is a classical and elaborate theory of conformal uniformization for domains in the Riemann sphere that are bounded by non-degenerate Jordan curves. Let Ω be such a domain which is also finitely connected. Koebe proved [42] that Ω is conformally homeomorphic to some domain Ω^* whose boundary components are circles. Such a domain is called a *circle domain*. Furthermore, Ω^* is unique up to Möbius transformations.

Discrete uniformization schemes have traditionally been the first step in constructing a sequence of approximations to a conformal map from the given domain (more on this in Section 0.1). There is much current interest and effort by (for example) Cannon, Floyd and Parry to provide sufficient conditions under which, discrete schemes based on the *discrete extremal length* method, will converge to a conformal map in the cases of triangulated annulus or a quadrilateral (see for instance [9, 10, 11, 12]). Of much current interest is the universality of the critical Ising and other lattice models where discrete complex analysis on graphs played crucial role (see for instance [15, 23]). However, it was shown by Schramm [50, page 117] that if one attempts to use the combinatorics of (for instance) the hexagonal lattice alone, square tilings (as constructed by Schramm's method) cannot be used as discrete approximations for the Riemann mapping.

In this paper, stemming from our work in [36, 37, 38], we will prove that a certain discrete scheme yields convergence of the mappings described below to a canonical *conformal* mapping from a given polygonal, planar, annulus, onto the interior of a Euclidean annulus bounded by two concentric circles.

Specifically, the underlying idea of this paper is rooted in a foundational feature of two dimensional conformal maps. If $f : \mathbb{D} \to \mathbb{C}$ is a conformal map, then the Cauchy-Riemann equations imply that $\Re(f)$ and $\Im(f)$ are harmonic functions, and that $\Im(f)$ is the harmonic conjugate of $\Re(f)$. For instance, when (r, θ) denote polar coordinates in the plane, we have that $u(r, \theta) = \log r$ and $v(r, \theta) = \theta$ (when θ is single valued) are harmonic functions, and $v(r, \theta)$ is the harmonic conjugate of $u(r, \theta)$. Indeed, in [38], starting with the solution g of a special discrete Dirichlet boundary value problem defined on a polygonal annulus we defined

two new functions on $\mathcal{T}^{(0)}$. The first function, g^*, which is actually defined for the annulus minus a *slit*[1], is obtained by integrating the discrete *normal derivative* of g along its level curves (see [38, Definition 2.6]). The second function, h, which depends on g^* is obtained via a solution of a discrete Dirichelt-Neumann boundary value problem on the domain of g^* (see [38, Definition 3.1]).

Hence, in the case of a planar triangulated annulus, and for any sequence of triangulations $\{\mathcal{T}_n\}$ of it, we have combinatorial approximating candidates for $u(r,\theta)$ (the g_n's), and two combinatorial candidates for $v(r,\theta)$ (the g_n^*'s or the h_n's). However, in this paper, due to the current state of the art in the L_∞ convergence results which we will apply, we will work with the pair $(g, \bar{g}^*)$ where $\bar{g}^*$ (see Definition 2.32) is a significant modification of g^*. In Theorem 3.12, we will show that, for a suitable sequence of quasi-uniform triangulations endowed with Stephenson's conductance constants defined according to Equation (1.2), the affinely extended maps of $\{g_n, \bar{g}_n^*\}$ will converge uniformly on compact subsets of a given annulus, to the real and imaginary parts of a conformal uniformizing map of the annulus, respectively. To this end, we will employ recent advances from the theory of the finite volume method, techniques from discrete potential theory, and classical results form the theory of functions of one complex variable and partial differential equations.

In order to put the needed advances over previous applications of the finite volume method in context, let us briefly recall an inspiring work by Dubejko [22]. Set w to be the solution of the Dirichlet problem $\Delta w = f$ for $x \in \Omega$, and $w = \phi$ for $x \in \partial\Omega$, where Ω is a simply-connected, bounded, Jordan domain with C^2 boundary, where $f \in L^2(\Omega)$ and $\phi \in C^0(\Omega)$. By applying techniques from the finite volume method, Dubejko proved that w can be approximated (in various norms) by a sequence of solutions of discrete Dirichlet boundary value problems. These solutions belong to a certain *Sobolev space* and are constructed via a sequence of triangulations (with special properties) that gets finer while exhausting Ω from the inside. Dubejko's work, which utilized Stephenson's conductance constants, is not sufficient for constructing approximations of conformal maps from Jordan domains. In fact, already in the simply connected case his techniques are not sufficient. This is due to the following reasons: First, his methods can be applied only under the assumption that the boundary of Ω is C^2; second, Dubejko did not address the problem of approximating a harmonic conjugate; third, Dubejko utilized Riemann's mapping theorem in his proof.

0.4. **Organization of the paper.** In Section 1, we describe the conductance constants suggested by Stephenson, and express these in the way applied in Theorem 3.12, the main theorem of this paper.

In Section 2, we present three novel definitions. We define the class of *discrete asymptotic harmonic functions*. Intuitively, a function in this class is *almost* harmonic on a scale determined by the mesh of the triangulation. This class will be employed in the approximation process described in the main theorem of this paper. The *flux following path* in a given triangulation is associated to the amount of *discrete flux* crossing a given path in the one skeleton of the Voronoi cells of the triangulation. Finally, utilizing a flux following path, we define a *conjugate function*, $\bar{g}^*$, of a discrete harmonic or a discrete asymptotic harmonic function.

[1]A slit is a chosen simple path in the annulus connecting the two boundary components on which g is monotone increasing.

In Section 3, we provide the proof of Theorem 3.12, where the convergence of our proposed discrete scheme is shown to converge in the case of a polygonal annulus. We are then concerned with the approximation of the uniformization of an annulus with continuous Jordan boundary. Corollary 3.38 demonstrates that Theorem 3.12 coupled with a generalization of a compactness theorem due to Koebe and a diagonalization process, allow the weakening of the boundary regularity assumption of Theorem 3.12 from polygonal to continuous.

Section 4 is devoted to the proof of Theorem 4.2, where we address the approximation of the uniformization of a bounded, simply-connected Jordan domain, the setting in which Conjecture 0.2 was first stated. The idea is to present the *punctured* domain as an increasing sequence of annuli. Thus, one can apply Theorem 3.12 to each annulus in the sequence. The existence of a converging subsequence of the maps obtained in each step to a bounded, conformal, univalent map follows the same rationale as in Corollary 3.38, and we can therefore restrict attention to the case that the boundary of the domain is polygonal. The final ingredient in the proof is the Riemann's removable singularity theorem.

With the aim of making this paper self-contained to a broad audience, it contains an Appendix. In Appendix A.1 and in Appendix A.2, we collect a few important notations, definitions and theorems from the finite volume method that are applied in this paper. The reader who is familiar with this method, can skip these sections. However, Theorem A.23, which is quoted from [14] is essential for the L_∞ convergence analysis results of this paper. In Appendix A.3, we explain how Stephenson's conductance constants naturally appear in the setting of the finite volume element method.

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1. Electrical networks induced by disk packings and Stephenson's conductances

Let us recall a few definitions and some notation from [21, 22, 54] and [55] in order to express the conductance constants suggested by Stephenson. Let P be a euclidean disk packing of a domain Ω for a complex K, i.e., the contact graph of P is isomorphic to K. For an interior edge $(u, v) \in K$, consider the tangent circles c_v, c_u as depicted in the figure below. Let c_x, c_y be their common neighboring circles.

The *radical center*, w_x, of the triple $\{c_v, c_u, c_x\}$ of circles will denote the center of the circle that is orthogonal to c_v, c_u and c_x and let w_y be the radical center of the triple $\{c_v, c_u, c_y\}$. Let z_u, z_v be the centers of c_u, c_v, respectively. Finally, for a vertex v, let R_v denote the radius of the circle z_v.

Stephenson's *conductance* of an edge is defined by (see also Definition A.10 and Equation (A.28)):

$$c(e) = c(u, v) = \frac{|w_x - w_y|}{|z_u - z_v|}. \tag{1.2}$$

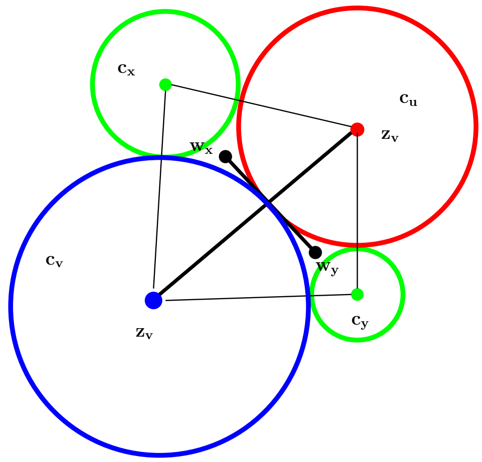

FIGURE 1.1. Constructing an edge conductance in a circle packing.

It is illuminating to give a probabilistic interpretation to this quantity. Stephenson's main idea was to chase angle changes at the centers of the circles, as radii changes while maintaining (new) disk packing. Given a euclidean circle packing, the effect of a small *increase* in the radius of one circle, say R_v, is that the angle sum Θ_v decreases, while the angle sums Θ_{v_j} at the neighboring vertices $\{v_1, v_2, \ldots, v_k\}$ *increase*. Some of the angle "distributed" by v arrives at v_j and must be passed along in order to keep a packing at v_j. Hence, R_{v_j} has to be adjusted and we need to keep track of the angle changes of its neighbors, and so forth.

In Euclidean geometry, the angles of any triangle add up to π, so angles in this process will never get lost. In other words, the total angle *leaving* one vertex must be divided into portions and then distributed as angles *arriving* to its neighbors. This movement can be expressed as a *Markov process*, where the transition probability from v to v_j, is the proportion of a *decrease* in angle at v that becomes an *increase* in angle at v_j. In this Markov process, the random walkers are the quantities of *angles* moving from one vertex to another. Thus, for a specific neighbor $u = v_j$, the amount of angle arriving at ψ_u is given by $\frac{d\psi_u}{dR_v}$; as a result, the transition probability from v to u as described above is given by

$$\bar{\rho}(v,u) = \frac{\frac{d\psi_u}{dR_v}}{\sum_{j=1}^{k} \frac{d\psi_{v_j}}{dR_{v_j}}}. \tag{1.3}$$

Also, for a vertex $v \in K$, we let

$$p(v,u) = \frac{c(v,u)}{\sum_{u \sim v} c(v,u)}. \tag{1.4}$$

It is remarkable that in 2005 (see [55, Section 18.5]) Stephenson showed that equality holds between these two Markov transitions, that is,

$$\rho(v,u) = \bar{\rho}(v,u), \; u \sim v. \tag{1.5}$$

2. Smooth harmonic conjugate functions and their combinatorial counterparts.

This section entails several key definitions and constructions. In the first subsection, we collect a few classical PDE existence results that go back to Poincaré and Lesbegue. In the second subsection, we will assume that $\mathcal{A}$ is a fixed, planar, polygonal annulus endowed with a triangulation $\mathcal{T}$. After recalling the definitions of the *combinatorial laplacian* and the *normal derivative*, we will turn to define the class of discrete, asymptotically harmonic functions. This class will enable us to define a conjugate function to a function which is either harmonic or asymptotically harmonic of some order (see Definition 2.18).

2.1. **Strong solutions of the Laplace equation and smooth harmonic conjugate functions.** Let Ω be a planar domain and assume that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is the *strong solution* of the Dirichlet boundary value problem for the Laplace equation

$$\begin{cases} \triangle u = 0 \text{ in } \Omega, \\ u = h \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where h is the trace of $\tilde{h} \in H^1(\Omega)$. It is well known that a strong solution is a also a *weak solution* (see Appendix A.2 or [16, Section 3.2]) for the details and the definition of $H^1(\Omega)$).

The study of the existence of strong solutions of Dirichlet boundary value type problems has an interesting history. Poincaré introduced the notion of *barriers*, and their importance was further recognized later by Lebesuge. A function $w \in C^0(\Omega)$ is called *super-harmonic* in Ω, if for any subdomain $\Omega' \subset \Omega$, and any harmonic function u in Ω',

$$w \geq u, \text{ in } \Omega'. \tag{2.2}$$

Let ξ be a point in $\partial\Omega$, then a $C^0(\bar{\Omega})$ function $w = w_\xi$ is called a *barrier* at ξ relative to Ω, if w is super-harmonic in Ω, it approaches 0 at ξ, and outside of any sphere about ξ, it has a positive lower bound in Ω. Two profound consequences of the existence of a barrier are the following.

Theorem 2.3 ([41, Theorem III, page 327])**.** *A necessary and sufficient condition that the Dirichlet problem for Ω is solvable for arbitrary assigned continuous boundary value values, is that a barrier for Ω exists at every point in $\partial\Omega$.*

Theorem 2.4 ([41, Theorem IV, page 328])**.** *Let u be the solution of the Dirichlet problem for Ω with $\phi \in C^0(\partial\Omega)$ boundary values. If $\xi \in \partial\Omega$ has a barrier, then*

$$\lim_{x \to \xi} u(x) = \phi(\xi). \tag{2.5}$$

It is therefore important to understand which domains in the Euclidean plan satisfy the hypothesis of Theorem 2.3. Indeed, general sufficient conditions can be described in terms of local properties of the boundary (see for instance [58, Proposition 5.13]).

Theorem 2.6 (Lebesgue)**.** *The Dirichlet boundary value problem (2.1) is solvable for arbitrary assigned continuous boundary value values if every component of the complement of the domain consists of more than a single point.*

For the applications of this paper, the following corollary is essential.

Corollary 2.7. *Let Ω be a Jordan domain, then the Dirichlet boundary problem (2.1) is solvable in Ω for arbitrary continuous boundary values.*

The (strong) maximum principle (see for instance, [16]) implies that a strong solution is unique. For $\Omega = \mathcal{A}$ a polygonal planar annulus, we can therefore make the following

Definition 2.8. We call $u \in C^2(\mathcal{A}) \cap C^0(\bar{\mathcal{A}})$ the strong solution of the Dirichlet boundary value problem of the Laplace equation, if

$$\begin{cases} \triangle u = 0 \text{ in } \mathcal{A}, \\ u = 1 \text{ on } E_1, \text{ and } u = 0 \text{ on } E_2. \end{cases} \tag{2.9}$$

We end this subsection by recalling the following definition which is valid for any harmonic function.

Definition 2.10 (A smooth harmonic conjugate ([46, Chapter 1.9])). Let (x_0, y_0) be a point in $\mathcal{A}$, and let (x, y) in $\mathcal{A}$ be an arbitrary different point. Let γ be a simple, counter-clockwise oriented, piecewise smooth curve joining (x_0, y_0) to (x, y) in $\mathcal{A}$. Let β be any simple, closed, counter-clockwise oriented, piecewise smooth curve in $\mathcal{A}$ whose winding number is equal to 1, and let s denote the arc-length parameter of these curves.

A harmonic conjugate of u is defined by

$$u^*(x, y) = u^*(x_0, y_0) + \int_\gamma \frac{\partial u}{\partial n} ds, \tag{2.11}$$

where $u^*(x_0, y_0)$ is some arbitrary fixed constant, and

$$\text{period}(u^*) = \int_\beta \frac{\partial u}{\partial n} ds. \tag{2.12}$$

Remark 2.13. It is well known that a smooth harmonic conjugate u^* is defined up to a constant, i.e., an assigned value at a point in the annulus; furthermore, the function values at any point differ by integral multiples of its period (see for instance [46, Chapter 1.9]).

2.2. **Discrete harmonic and asymptotically harmonic functions, and their conjugates.** We now turn to defining a combinatorial function analogous to u^*. We will start with some notation and definitions from the subject of discrete harmonic analysis that will be used throughout the rest of this paper (see for instance [5] or [38, Section 1.1]). Let $\Gamma = (V, E, c)$ be a planar *finite network*; that is, a planar, simple, and finite connected graph with vertex set V and edge set E, where each edge $(x, y) \in E$ is assigned a *conductance* $c(x, y) = c(y, x) > 0$. Let $\mathcal{P}(V)$ denote the set of non-negative functions on V. Given $F \subset V$, we denote by F^c its complement in V. Set $\mathcal{P}(F) = \{u \in \mathcal{P}(V) : S(u) \subset F\}$, where $S(u) = \{x \in V : u(x) \neq 0\}$. The set $\delta F = \{x \in F^c : (x, y) \in E \text{ for some } y \in F\}$ is called the *vertex boundary* of F. Let $\bar{F} = F \cup \delta F$, and let $\bar{E} = \{(x, y) \in E : x \in F\}$. Let $\bar{\Gamma}(F) = (\bar{F}, \bar{E}, \bar{c})$ be the network such that $\bar{c}$ is the restriction of c to $\bar{E}$. We write $x \sim y$ if $(x, y) \in \bar{E}$. The following operators are discrete analogues of classical notions in continuous potential theory (see for instance [26] and [17]).

Definition 2.14. Let $u \in \mathcal{P}(\bar{F})$. Then for $x \in F$, the function

$$\Delta u(x) = \sum_{y\sim x} c(x,y)\,(u(x) - u(y)) \tag{2.15}$$

is called the *Laplacian* of u at x. For $x \in \delta(F)$, let $\{y_1, y_2, \ldots, y_m\} \in F$ be its neighbors enumerated clockwise. The *normal derivative* of u at a point $x \in \delta F$ with respect to a set F is defined by

$$\frac{\partial u}{\partial n}(F)(x) = \sum_{y\sim x,\ y\in F} c(x,y)(u(x) - u(y)). \tag{2.16}$$

Finally, $u \in \mathcal{P}(\bar{F})$ is called *harmonic* in $F \subset V$ if $\Delta u(x) = 0$, for all $x \in F$.

In order to define the class of discrete, asymptotically harmonic functions, we will use notation and definitions from Appendix A.1; in particular, *every* triangulation in this section is assumed to be quasi-uniform and consisting solely of *nonobtuse* triangles (see condition (V0) in Appendix A.1).

Given such a triangulation $\mathcal{T}$ of $\mathcal{A}$, following [30, Chapter 2], for each one of its Voronoi cells Ω_i which is centered at x_i, we let

$$\lambda_i = \left(\max_{j\in N_i} l(\Gamma_{i,j})\right)^{1/2} \text{ and } \lambda = \max_{i\in J} \lambda_i, \tag{2.17}$$

where $l(\cdot)$ denotes Euclidean length.

Definition 2.18. Let $\alpha \in \mathbb{R}$ be a positive constant. Let $\mathcal{T}$ be a triangulation of $\mathcal{A}$, with Voronoi cells $\{\Omega_i\}_{i\in J}$. A function $g : \mathcal{T}^{(0)} \to \mathbb{R}$ is said to be asymptotically harmonic of order α with respect to conductance constants $\{c_{i,j} = c(x_i, x_j)\}$, if for all $i \in J$ we have

$$\big|\Delta(x_i)\big| = \Big|\sum_{j\in N_i} c(i,j)\big(g(x_j) - g(x_i)\big)\Big| = O(\lambda^\alpha). \tag{2.19}$$

The following lemma provides an estimate for the integral of the normal derivative of g, which can be thought of as the *discrete flux* of g through the boundary of a Voronoi cell.

Lemma 2.21 (Asymptotic flux estimate)**.** *Let $g : \mathcal{T}^{(0)} \to \mathbb{R}$ be harmonic or asymptotically harmonic of order α, with respect to conductance constants $\{c_{i,j} = c_{[x_i,x_j]}\}$. Then, for any homotopically trivial (in $\mathcal{A}$), closed path $\gamma \subset \mathcal{T}^{(1)}$ which contains an integer number of Voronoi cells Ω_i, $i \in J$, we have*

$$\int_{x\in\gamma} \frac{\partial g}{\partial n}(x) = \begin{cases} 0, & \textit{if } g \textit{ is harmonic,} \\ O(\lambda^\alpha), & \textit{if } g \textit{ is asymptotically harmonic or order } \alpha. \end{cases} \tag{2.22}$$

Proof. Let $\Omega_m = \cup_{i=1}^m \Omega_i$ be the maximal collection of control volumes enclosed in γ, and let E_m be those edges of $\mathcal{T}^{(1)}$ that lies in the interior of the bounded region enclosed by γ. The first Green identity (see for instance [5, Proposition 5]) implies that for $u, v \in \mathcal{P}(\Omega_m)$, we have

(2.23)

$$\int_{[i,j]\in \bar{E}_m} c(i,j)\big(u(i) - u(j)\big)\big(v(i) - v(j)\big) = \int_{x\in\Omega_m\cap\mathcal{T}^{(0)}} \Delta u(x)v(x) + \int_{y\in\gamma} \frac{\partial u}{\partial n}(\Omega_m)(y)v(y).$$

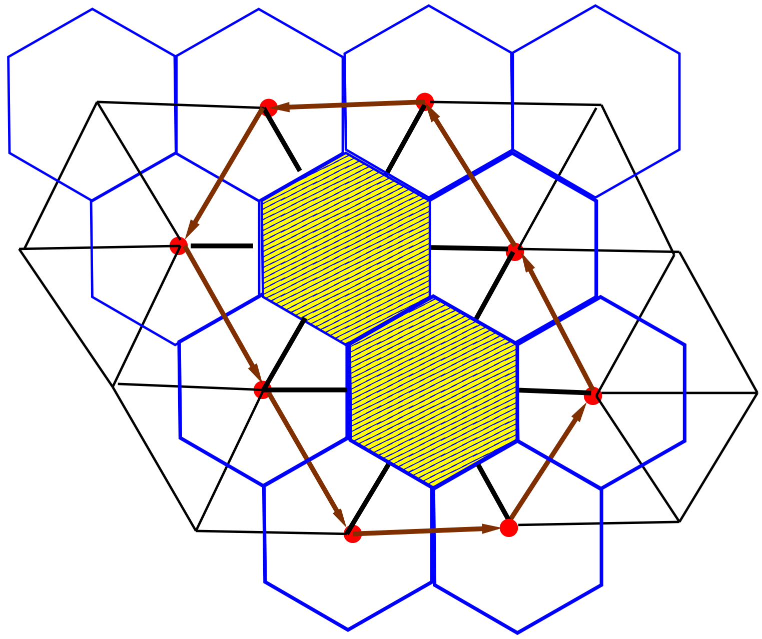

FIGURE 2.20. An illustration of γ (solid brown) for Lemma 2.21.

We now let $v \equiv 1$ in the above equality, and obtain

$$0 = \sum_{i=1}^{m} \Delta u(x_i) + \int_{y\in\gamma} \frac{\partial u}{\partial n}(\Omega_m)(y). \tag{2.24}$$

It therefore follows, by the definition of the combinatorial laplacian, that

$$0 = \sum_{i=1}^{m} \sum_{j\in N_i} c(i,j)\big(u(j) - u(i)\big) + \int_{y\in\gamma} \frac{\partial u}{\partial n}(\Omega_m)(x), \tag{2.25}$$

and the assertions of the lemma readily follow.

□

The following Corollary immediately follows.

Corollary 2.26 (Asymptotic path independence)**.** *Let γ_1 and γ_2 be two simple paths in $\mathcal{T}^{(1)} \subset \mathcal{A}$ joining two vertices $v_1, v_2 \in \mathcal{T}^{(0)}$, such that the path $\gamma_2^{-1} \circ \gamma_1$ is trivial in $\pi_1(\mathcal{A})$, and contains an integer number of control volumes Ω_i. Then*

(2.27)

$$\int_{x\in\gamma_1} \frac{\partial g}{\partial n}(x) - \int_{x\in\gamma_2} \frac{\partial g}{\partial n}(x) = \begin{cases} 0, & \text{if } g \text{ is harmonic,} \\ O(\lambda^{\alpha}), & \text{if } g \text{ is asymptotically harmonic or order } \alpha. \end{cases}$$

Let g be a discrete harmonic or asymptotically harmonic function. Before defining the notion of a *combinatorial conjugate* function for g, we will need to define a special class of paths in $\mathcal{T}^{(1)}$. Thereafter, by integrating a generalized version of the normal derivative of g,

along a path from this class, the combinatorial conjugate function of g will be defined at the vertices of the Voronoi cells of $\mathcal{T}$.

We let Λ denote the union of all Voronoi cells of $\mathcal{T}$, and make

Definition 2.28 (Flux fellow paths)**.** Let ω_0 be a fixed vertex in a Voroni cell of $\mathcal{T}$ and let ω be any vertex in Λ. Let $\gamma_\Lambda = [\omega_0, \ldots, \omega_k = \omega]$ be a simple, piecewise linear curve in $\Lambda^{(1)}$ joining ω_0 to ω. For each $[\omega_i, \omega_{i+1}]$, $i = 0, \ldots, k-1$, let v_i be the vertex in $\mathcal{T}^{(0)}$ on the unique edge intersecting $[\omega_i, \omega_{i+1}]$, and which is to the right of $[\omega_i, \omega_{i+1}]$. Then $\gamma_{\mathcal{T}} = [v_0, \ldots, v_{k-1}] \subset \mathcal{T}^{(1)}$ will be called the *flux fellow path* of γ_Λ.

Remark 2.29. The discussion preceding condition (V0) in Appendin A.1 grants us that $\gamma_{\mathcal{T}}$ is indeed a path in $\mathcal{T}^{(1)}$; we orient each edge in γ_Λ according to increasing order of its vertices.

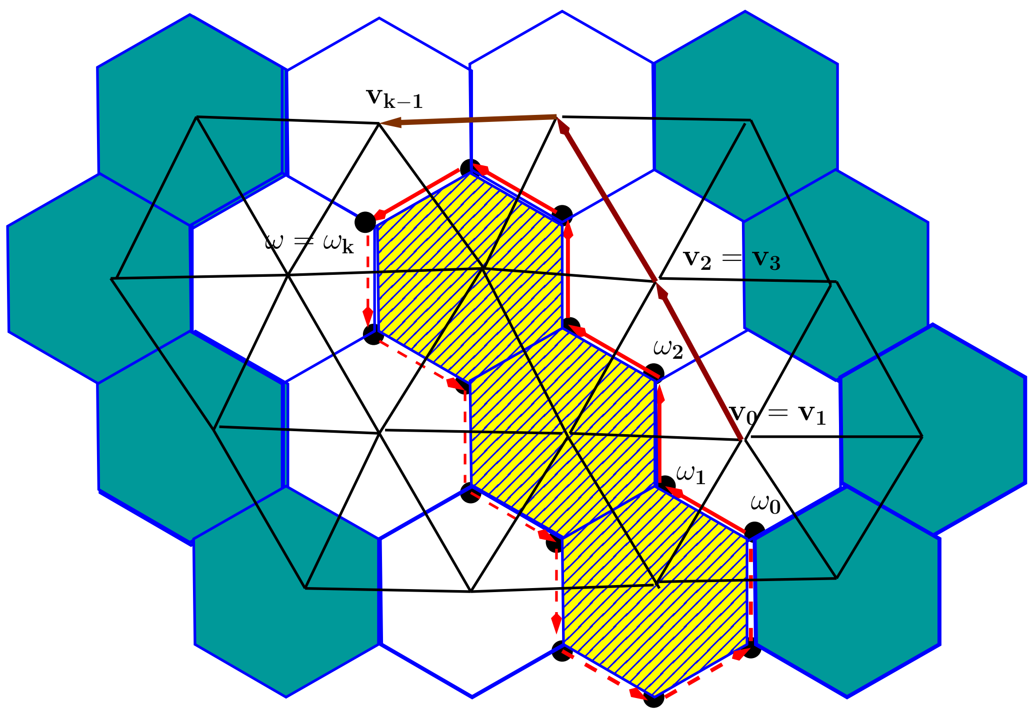

FIGURE 2.30. The path γ_Λ (solid red) and $\gamma_{\mathcal{T}}$ (solid brown).

With $\omega_0, \omega, \gamma_\Lambda$ and $\gamma_{\mathcal{T}}$ defined as above, we extend γ_Λ in $\Lambda^{(1)}$ to a simple, piecewise linear and homotopically trivial, closed curve in $\mathcal{A}$, in such a way that the piecewise linear disk bounded by it, and $\gamma_{\mathcal{T}}$ are disjoint. We let $F(\omega_0, \omega, \gamma_\Lambda, \gamma_{\mathcal{T}})$ denote such a disk. Note that γ_Λ and $F(\omega_0, \omega, \gamma_\Lambda, \gamma_{\mathcal{T}})$ are not uniquely determined; however, once γ_Λ is chosen, $\gamma_{\mathcal{T}}$ is uniquely determined. (In Figure 2.30, $F(\omega_0, \omega, \gamma_\Lambda, \gamma_{\mathcal{T}})$ is the union of three Voronoi cells.)

We now need to extend the notion of the discrete normal derivative (see Equation (2.16) in Definition 2.14) in order to treat the special case arising in the definition above. Specifically, vertices in which the formally defined normal derivative needs to be computed do *not* belong to the vertex boundary of the domain, rather, they are close to it (in a combinatorial sense) as is the case with vertices that belong to $\gamma_{\mathcal{T}}$ and γ_Λ, respectively. In the definition that

will follow, we will abuse notation and keep the notation in (2.16) for the extension. The extension needed for the applications of this paper is determined by the following

Convention. The computation of the integrand in

$$\int_{y\in\gamma_{\mathcal{T}}} \frac{\partial g}{\partial n}(F(\omega_0,\omega,\gamma_\Lambda,\gamma_{\mathcal{T}}))(y), y \in \gamma_{\mathcal{T}}, \tag{2.31}$$

will include only vertices that are adjacent to y along an edge which intersects γ_Λ.

It is time to make the third, and final key definition of this section.

Definition 2.32 (A combinatorial conjugate)**.** Let $\mathcal{T}$ be a triangulation of $\mathcal{A}$, and let Λ denote the union of all Voronoi cells of $\mathcal{T}$. Let ω_0 be a fixed vertex in Λ and let ω be any vertex in Λ, let γ_Λ be a simple, piecewise linear curve in Λ joining ω_0 to ω, let $\gamma_{\mathcal{T}}$ be the flux fellow path of γ_Λ, and let $F(\omega_0,\omega,\gamma_\Lambda,\gamma_{\mathcal{T}})$ be a disk as described above.

(i) Let g be a discrete harmonic, or a discrete asymptotically harmonic function of order α. Then, a combinatorial conjugate of g is defined by

$$\bar{g}^*(\omega) = \bar{g}^*(\omega_0) + \int_{y\in\gamma_{\mathcal{T}}} \frac{\partial g}{\partial n}(F(\omega_0,\omega,\gamma_\Lambda,\gamma_{\mathcal{T}}))(y) \text{ for every } \omega \in \Lambda, \tag{2.33}$$

where $\bar{g}^*(\omega_0)$ is some arbitrary, fixed constant.

We divide each 2-cell in $\Lambda^{(2)}$ into triangles with vertices in $\Lambda^{(0)}$ and disjoint interiors. We then extend g^* affinely over edges in $\Lambda^{(1)}$ and triangles in $\Lambda^{(2)}$. (The extended function will also be called the combinatorial conjugate of g.)

(ii) Let α_Λ be any simple, counter-clockwise oriented, closed curve in Λ whose winding number is equal to 1. The period of $\bar{g}^*$ is defined by

$$\text{period}(\bar{g}^*) = \int_{\xi\in\alpha_{\mathcal{T}}} \frac{\partial g}{\partial n}(\xi). \tag{2.34}$$

Remark 2.35. A notion of combinatorial period for g^* was introduced in [38, Section 2], and combinatorial provisions analogous to those in Remark 2.13 hold for $\bar{g}^*$. In addition, it is also clear that the definition is independent of the choice of $F(\omega_0,\omega,\gamma_\Lambda,\gamma_{\mathcal{T}})$, and of the choice of γ_Λ due to Corollary 2.26.

Remark 2.36. In the definition of $\bar{g}^*$, the main new differences compared to [38, Definition 2.6] are: (i) it incorporates the Voroni cells, and (ii) the integration of the normal derivative of g is computed along flux fellow paths. These are needed since in the applications of this paper, the combinatorial function analogous to u is *asymptotically harmonic*, and not harmonic as was the case in [38].

Remark 2.37. The search for discrete approximation of conformal maps has a long and rich history. We refer to [45] and [15, Section 2] as excellent recent accounts. We should also mention that a search for a combinatorial Hodge star operator has recently gained much attention and is closely related to the construction of a harmonic conjugate function. We refer the reader to [40] and to [47] for further details and examples for such combinatorial operators.

3. Uniformization of a planar annulus

In this section, we prove the main theorem of this paper, Theorem 3.12, and afterwards, we will indicate how the hypotheses "polygonal boundary" in the main theorem, can be relaxed to "continuous boundary".

3.1. **Uniformization of a planar polygonal annulus.** In this subsection, we will continue to assume that $\mathcal{A}$ is a fixed polygonal annulus. We will construct a sequence of mappings, obtained via a refined sequence of quasi-uniform triangulations and conductance constants along edges according to (3.13), from the interior of $\mathcal{A}$ onto the interior of a concentric Euclidean annulus. The dimensions of the image annulus are determined (see Equation (3.16)) by a specific Dirichelt boundary value problem. Theorem 3.12 demonstrates that the sequence converges uniformly on compact subsets to a conformal homeomorphism.

We keep the notation of the previous sections and Appendices (A)-(C). In particular, let $\mathcal{A}$ be endowed with a family of quasi-uniform triangulations $\{\mathcal{T}_{\rho_n}\}$ such that $\rho_n \to 0$, as $n \to \infty$. For each $\mathcal{T}_{\rho_n}$, let the corresponding family of Voronoi cells be denoted by $\{\Omega_n = \Omega_{\rho_n}\}$. In addition to requiring that each triangulation is quasi-uniform, we will assume the existence of a constant τ_0 such that for all $\rho_n < \tau_0$, the corresponding family of Voronoi cells Ω_n satisfies the following assumptions:

(V1): The number of essential neighbors of each v_i remains uniformly bounded, that is,

$$\max_{i\in J}\{\mathrm{card}N_i\} \leq m_* \text{ for some } \ m_* \in \mathbb{N};$$

(V2): Each point $x_{i,j} = [x_i, x_j] \cap \Gamma_{i,j}$ is the middle point of the segment $\Gamma_{i,j}$.

For $W \subset \mathcal{A}$ and two points $a, b \in W$, we let $D(a, b)$ denote the pseudo-distance on W defined as the infimum of the Euclidean lengths of curves in W that join a to b. We then define the *intrinsic diameter* of Ω by

$$\mathrm{IDiam}(W) = \sup\{D(a, b) \mid a, b \in W\}. \tag{3.1}$$

In Lemma 3.5, we will need to consider Dirichlet boundary value problems with prescribed Poisson data and non-homogeneous boundary condition as described in the following

Definition 3.2. Let $h : \partial\mathcal{A} \to \mathbb{R}$ be a continuous function and let $\tilde{h}$ be a $C^4(\mathcal{A})\cap C^0(\mathcal{A}\cup\partial\mathcal{A})$ extension of h. A function $\tilde{u}$ in $C^2(\mathcal{A})\cap C^{(0)}(\partial\mathcal{A})$ is a (strong) solution of a Dirchlet boundary value problem for the Poisson equation and non-homogeneous boundary data, if for $f = -\Delta\tilde{h}$ and for $g = \tilde{h}|\partial\mathcal{A}$, we have

$$\begin{cases} \Delta\tilde{u} = f \text{ in } \mathcal{A}, \text{ and} \\ \tilde{u} = g, \text{ on } \partial\mathcal{A}. \end{cases} \tag{3.3}$$

Remark 3.4. The existence and uniqueness of a strong solution to (3.3) follows from Corollary 2.7, by setting $\tilde{u} = u - \tilde{h}$, where u is the strong solution of (2.1).

Several ideas from the proof of the following lemma which provides an estimate for the discrete flux of a solution of (3.3) along the full boundary of one Voronoi cell, will be essential in the proof of Theorem 3.12. We will apply a weaker version of the following

Lemma 3.5 ([30, Lemma 2.63]). *Assume that conditions (V1) and (V2) hold. Let the solution of (3.3) belong to* $C^4(\bar{\Omega})$. *Then there exists some constant* $c = c(\tilde{u}, \Omega)$ *such that*

$$\left| \sum_{j\in N_i} \frac{m_{i,j}}{d_{i,j}} \big(\tilde{u}(x_j) - \tilde{u}(x_i)\big) - \int_{\Omega_i} f dx \right| \leq c\lambda_i^3, \text{ for all } i \in J, \tag{3.6}$$

where λ_i *is defined by (2.17).*

Remark 3.7. Assumptions (V1) and (V2) are critical in the proof of this Lemma. In the statement of this Lemma, Ω is the domain on which (3.3) is initially defined. However, the proof remains valid even if the regularity assumption is only assumed to hold for any close, proper subset of Ω with *thick* enough neighborhood; that is, if the subset and its neighborhood are still contained in Ω. This weaker assumption will be used in the proof of Theorem 3.12 below, where we will also show how to choose a thick neighborhood.

Let us now restrict our attention to the particular type of boundary value problem (2.9) that will be considered in Theorem 3.12. Let h be the continuous function defined on $\partial\mathcal{A}$ by setting

$$h|E_1 = 1 \text{ and } h|E_2 = 0. \tag{3.8}$$

Let $\tilde{h} \in C^4(\mathcal{A}) \cap C^0(\partial\mathcal{A})$ be an extension of h to the interior of $\mathcal{A}$. Let us denote the projection of $\tilde{h}$ on Ω_n by $\Pi_n(\tilde{h})$, that is, we first define

$$\Pi_n(\tilde{h})(x) = \tilde{h}(x), \text{ for every } x \in \Omega_n^{(0)}, \tag{3.9}$$

and then extend affinely over triangles.

Let u be a solution of (2.9). Recall that $u = \tilde{u} + \tilde{h}$, where the *trace* of $\tilde{h} \in L^2(\mathcal{A})$ is equal to the trace of u on $\partial\mathcal{A}$, and $\tilde{u}$ is the solution of the homogeneous Dirichlet boundary value problem for Poisson's equation

$$\begin{cases} \Delta\tilde{u} = -\Delta\tilde{h}, \text{ in } \mathcal{A} \\ \tilde{u} = 0, \text{ on } \partial\mathcal{A}. \end{cases} \tag{3.10}$$

Definition 3.11. Let $\tilde{u}$ and $\tilde{u}_n = \tilde{u}_{\rho_n}$ be the solutions of (A.1), presented as in (A.22) with $f = -\Delta\tilde{h}$, respectively.

With the notation above, we now turn to the main theorem of this paper. In the statement below, quasi-uniform triangulations are the subject of Definition A.7; Definition A.10 and Equation (A.28) explain the terms appearing in the definition of the conductances, Equation (3.13), and exploits the connection to Stephenson's conjecture (Conjecture 0.2). In order to ease the notation, we will not distinguish between a map defined on the 0-skeleton of a triangulation and the affine extension of the map on edges and triangles. Finally, for every n, let $\mathcal{T}_n = \mathcal{T}_{\rho_n}$.

Theorem 3.12. *Let $\{\mathcal{T}_n\}$ be a sequence of quasi-uniform triangulations of $\mathcal{A}$ of mesh size $\rho_n \to 0$ as $n \to \infty$, and let the corresponding family of Voronoi cells of each $\mathcal{T}_n$ be denoted by $\{\Omega_n\}$. Assume in addition that $\{\mathcal{T}_n\}$ satisfies conditions $(V1)$ and $(V2)$. Let the conductance of each edge $e \in T$, $T \in \mathcal{T}_n^{(1)}$ be defined by*

$$c_n(e) = \frac{m^T_{ij,n}}{d_{ij,n}}. \tag{3.13}$$

Let u and h be given in (2.9) and (3.8), respectively, and define (see Definition (3.11))

$$g_n = \tilde{u}_n + \Pi_n(\tilde{h}). \tag{3.14}$$

Then, as $n \to \infty$ the following assertions hold:

(1) $\|u - g_n\|_{L_\infty(\mathcal{A})} \to 0$.
(2) *On each proper, compact subset of $\mathcal{A}$, the g_n's are asymptotically harmonic of order $\alpha = 3$.*
(3) *Let $\bar{g}^*_n$ denote the combinatorial conjugate of g_n, and let ϕ_n be the sequence of discrete mappings defined by extending affinely the discrete sequence of mappings given by*

$$\phi_n(\omega) = \exp\big(\frac{2\pi}{\mathrm{period}(\bar{g}^*_n)}\big(g_n(\omega) + i\bar{g}^*_n(\omega)\big)\big), \ \omega \in \mathcal{A} \cap \Omega_n^{(0)}, \tag{3.15}$$

over Ω_n. Then the sequence $\{\phi_n\}$ converges uniformly on compact subsets of $\mathcal{A}$ to a conformal homeomorphism, denoted by $\Psi_{\mathcal{A}}$, onto the interior of the concentric Euclidean annulus $\mathcal{E}_{\mathcal{A}}$, whose inner and outer radii are given by

$$\{R_1, R_2\} = \{1, \exp\big(\frac{2\pi}{\mathrm{period}(u^*)}\big)\}, \tag{3.16}$$

where u^* and period(u^*) are given in Definition 2.10.

Remark 3.17. Following this procedure for every $\rho > 0$, we have turned $\mathcal{T}_\rho$ into a finite electrical network, as predicted in Stephenson's conjecture (see Conjecture 0.2), where the induced potential function defined on it (see [52]) satisfies the system of equations described by (A.31). We remark that since for each $\rho > 0$, the values of u_ρ at the boundary vertices $\partial\mathcal{T}^0 \subset \partial\Omega$ are given, there is no need to specify conductance constants for edges that are contained in $\partial\Omega$; or one can choose arbitrary values.

Proof. Setting $f = -\Delta\tilde{h}$ in Theorem A.23, we know that for the approximation of $\tilde{u}$ by $\tilde{u}_n$ (see Equation (3.10)), the following estimate holds. There exist constants $C = C(\mathcal{A})$ and a real number $s = s(\mathcal{A}) \in (0, 1)$ such that

$$\|\tilde{u} - \tilde{u}_n\|_{L_\infty(\mathcal{A})} \le C\rho_n^s \log(1/\rho_n)\|\Delta\tilde{h}\|_{L_p(\mathcal{A})}. \tag{3.18}$$

In particular, as $\rho_n \to 0$ we have that

$$\|\tilde{u} - \tilde{u}_n\|_{L_\infty(\mathcal{A})} \to 0. \tag{3.19}$$

Since both $\tilde{h}$ and $\Pi_n(\tilde{h})$ are continuous in $\mathcal{A}$, we also have that

$$\|\tilde{h} - \tilde{\Pi}_n(\tilde{h})\|_{L_\infty(\mathcal{A})} \to 0. \tag{3.20}$$

Hence, we are now able to show that the g_n's comprise our desired approximations to u - the strong solution of the smooth Dirichlet problem for the Laplace equation (2.9). Precisely, by recalling Remark 3.4, we observe that

$$
\begin{aligned}
\|u - (\Pi_n(\tilde{h}) + \tilde{u}_n)\|_{L_\infty(\mathcal{A})} &= \|u - \tilde{h} + \tilde{h} - (\tilde{u}_n + \Pi_n(\tilde{h}))\|_{L_\infty(\mathcal{A})} \\
&= \|\tilde{u} + \tilde{h} - (\tilde{u}_n + \Pi_n(\tilde{h}))\|_{L_\infty(\mathcal{A})} \\
&= \|(\tilde{u} - \tilde{u}_n) + (\tilde{h} - \Pi_n(\tilde{h}))\|_{L_\infty(\mathcal{A})} \\
&\leq \|\tilde{u} - \tilde{u}_n\|_{L_\infty(\mathcal{A})} + \|\tilde{h} - \Pi_n(\tilde{h})\|_{L_\infty(\mathcal{A})}.
\end{aligned} \tag{3.21}
$$

Therefore, the first assertion of the Theorem is proved by applying Equations (3.19) and (3.20).

We will continue the proof by showing that for n large enough, each g_n is asymptotically harmonic of the same order. To this end, recall that, by definition, for every vertex $v_i \in \Omega_n^{(0)}$, we have

$$\Pi_n(\tilde{h})(v_i) = \tilde{h}(v_i). \tag{3.22}$$

It is clear that $\tilde{h}$ is a solution of (3.3) with $g = \tilde{h}|\partial\mathcal{A}$ and with $f = \Delta\tilde{h}$. Therefore, for each n, applying Equation (3.6) with $h_n = \Pi_n(\tilde{h})$ and $f = \Delta\tilde{h}$, together with taking $\rho = \rho_n$, and $u_\rho = \tilde{u}_n$ in Equation (A.31), imply that each g_n is discrete asymptotically harmonic of order $\alpha = 3$. By construction, it also holds that $g_n|\mathcal{T}_n^{(0)} \cap E_1 = 1$ and $g_n|\mathcal{T}_n^0 \cap E_2 = 0$, for all n.

We now turn to prove the uniform convergence of the $\bar{g}_n^*$'s, over compact subsets of $\mathcal{A}$, to u^*- the harmonic conjugate of u. Indeed, let $\mathcal{A}^\epsilon \subsetneq \mathcal{A}$ be a compact annulus with smooth boundary which is concentric with $\mathcal{A}$, where $\epsilon = \text{dist}(\partial\mathcal{A}, \partial\mathcal{A}^\epsilon)$ is small. Let us choose n large enough (i.e., ρ_n small enough) so that there exists a triangulation $\mathcal{T}_{\rho_n} \in \{\mathcal{T}_n\}$ satisfying the following.

- **(V3):** A subset of its associated volume elements $\{\Omega_{i,\rho_n}\}_{i\in J'}$ is contained in $\mathcal{A}^\epsilon$, and the combinatorial one vertex neighborhood of this subset, when considered in $\mathcal{T}_{\rho_n}^{(0)}$, is also contained in $\mathcal{A}^\epsilon$.

Following Definition 2.28 and the discussion proceeding it, we choose an index $i \in J'$, as well as one of the vertices of Ω_{i,ρ_n}, which will be denoted by ω_0. Let ω be any vertex in $\cup_{i\in J'}\Omega_{i,\rho_n}$. Let $\gamma_{[\omega_0,\omega]}^{\rho_n} = [\omega_0, \omega_1, \ldots, \omega_{k-1}, \omega = \omega_k]$ be a (piecewise linear) simple, counter-clockwise oriented path in the one skeleton of $\Lambda_{\rho_n} = \cup_{i\in J'}\Omega_{i,\rho_n}$, joining ω_0 to ω. We will also define $u^*(\omega_0) = 0$.

It then follows from Definition 2.10 and Remark 2.13, that the smooth harmonic conjugate function u^* satisfies

$$u^*(\omega) = \int_{\gamma_{\Lambda_{\rho_n}}[\omega_0,\omega]} \frac{\partial u}{\partial n} ds. \tag{3.23}$$

We now follow the notation in Definition 2.28, and we let $\gamma_{\mathcal{T}_{\rho_n}}$ denote the flux fellow path of $\gamma_{\Lambda_{\rho_n}}[\omega_0, \omega]$. Let us write $\gamma_{\mathcal{T}_{\rho_n}} = [v_0^{\rho_n}, \ldots, v_{k-1}^{\rho_n}]$, and set $\bar{g}_n^*(\omega_0) = 0$. Hence, by

Definition 2.32, the asymptotic combinatorial conjugate of $g_n = g_{\rho_n}$ is defined at ω by

$$\bar{g}_n^*(\omega) = \int_{x\in\gamma_{\mathcal{T}_{\rho_n}}} \frac{\partial g_n}{\partial n}(x). \tag{3.24}$$

We now turn to estimate

$$|u^*(\omega) - \bar{g}_n^*(\omega)| = \Big| \int_{\gamma_{\Lambda_{\rho_n}}[\omega_0,\omega]} \frac{\partial u}{\partial n} ds - \int_{x\in\gamma_{\mathcal{T}_{\rho_n}}} \frac{\partial g_n}{\partial n}(x) \Big|. \tag{3.25}$$

In fact, we are interested in this difference as $n \to \infty$. Hence, by the extension of the definition of the combinatorial normal derivative (see Equation (2.16) and Equation (2.31)), and the first assertion of the theorem, it is sufficient to replace g_n, in any term appearing in the second integrand in (3.25), with the restriction of u to the vertices in $\mathcal{T}_{\rho_n}$.

It is classical that $u \in C^4(\mathcal{A}^\epsilon)$, and therefore Equation (5.11) in [30] shows that for $x_i = x_{i,\rho_n}, x_j = x_{j,\rho_n}$, $\Gamma_{i,j} = [\omega_i, \omega_j]$ and with $\lambda_i = \lambda_{i,\rho_n}$ (see (2.17)), there exists a positive constant $c_0 = c_0(u, \mathcal{A}^\epsilon)$ so that

$$\Big| \frac{1}{d_{i,j}} \big(u(x_j) - u(x_i)\big) - \frac{\partial u}{\partial n}_{i,j}(x_{i,j}) \Big| \leq c_0 \lambda_i^3,\ \ i \in J', j \in N_i. \tag{3.26}$$

Equation (5.15) in [30] shows that for each $i \in J'$, the following continuous linear functional T_i (which can actually be defined for any function in $C^3(\bar{\Omega}_{i,\rho_n})$)

$$T_i(u) = \sum_{j\in N_i} \Big(\int_{\Gamma_{i,j}} \frac{\partial u}{\partial n}_{i,j} ds - m_{i,j} \frac{\partial u}{\partial n}_{i,j}(x_{i,j}) \Big), \tag{3.27}$$

the following estimate holds

$$|T_i u| \leq c_1 \lambda_i^3 \tag{3.28}$$

where $c_1 = c_1(u, \Omega_{i,\rho_n})$.

It therefore follows that

(3.29)

$$\begin{aligned} \Big| \int_{\gamma_{\Lambda_{\rho_n}}[\omega_0,\omega]} \frac{\partial u}{\partial n} ds - \int_{x\in\gamma_{\mathcal{T}_{\rho_n}}} \frac{\partial g_n}{\partial n}(x) \Big| \ &\approx\ \Big| \sum_{i=0}^{k} \Big(\int_{[\omega_i,\omega_{i+1}]} \frac{\partial u}{\partial n}_{i,i+1} ds - m_{i,j} \frac{\partial u}{\partial n}_{i,j}(x_{i,j}) \Big) \Big| \\ &\leq\ \sum_{i=0}^{k} \Big| \Big(\int_{[\omega_i,\omega_{i+1}]} \frac{\partial u}{\partial n}_{i,i+1} ds - m_{i,j} \frac{\partial u}{\partial n}_{i,j}(x_{i,j}) \Big) \Big| \\ &\leq\ \frac{\mathrm{IDiam}(\mathcal{A}^\epsilon)}{\lambda} c_2(u, \mathcal{A}^\epsilon) \lambda_i^3 \\ &\leq\ c_3(u, \mathcal{A}^\epsilon) \lambda^{-1} \lambda_i^3 \\ &\leq\ c_3 \lambda^2. \end{aligned}$$

It is evident that as $n \to 0$ we have that $\lambda \to 0$.

We now study several choices that are to be made as we let $n \to \infty$, and as a result the triangulation is changing. Let $m > n$, and let $\mathcal{T}_m = \mathcal{T}_{\rho_m}$ be the corresponding triangulation

which we may again assume satisfies condition (V3) with the appropriate indices changes. Following the paragraph preceding (V3), we let q_0 and q be the analogous choices for ω_0 and ω, respectively. Let σ be any smooth path in $\mathcal{A}^\epsilon$ joining ω_0 to q_0, we then define

$$\tau = \tau(\sigma) = \int_\sigma \frac{\partial u}{\partial n} ds, \tag{3.30}$$

and accordingly set

$$g_m^*(q_0) = \tau. \tag{3.31}$$

Then, it readily follows that an estimate analogous to Equation (3.29), holds for the difference

$$|u^*(q) - \bar{g}_m^*(q)|. \tag{3.32}$$

Hence, we conclude that

$$\lim_{n\to\infty} \sup_{x \in \cup_i \Omega_{i,\rho_n}^{(0)}} |u^*(x) - \bar{g}_n^*(x)| = 0, \tag{3.33}$$

and as $n \to \infty$, $\cup_i \Omega_{i,\rho_n}^{(0)}$ with $i \in J'(\rho_n)$ comprises a dense subset of $\mathcal{A}^\epsilon$. Therefore, applying the uniform continuity of the $\bar{g}_n^*$'s and u^* in $\mathcal{A}^\epsilon$ and letting $\epsilon \to 0$, we obtain uniform convergence of the $\bar{g}_n^*$'s to u^* over compact subsets of $\mathcal{A}$.

We also need to prove that

$$\text{period}(\bar{g}_n^*) \to \text{period}(u^*). \tag{3.34}$$

To this end, let us choose a point P_0 in $\mathcal{A}^\epsilon$, and let β and period(u^*) be given according to Definition 2.10. Furthermore, let $\omega_n \in \Lambda_n$ be chosen so that $\omega_n \to P_0$. Let $\gamma_{\mathcal{T}_n}$ be a closed curve in $\Lambda_n^{(1)}$, based at ω_n according to which period$(\bar{g}_n^*)$ is computed.

Since u^* is continuous in $\mathcal{A}^\epsilon$, we have that

$$u^*(\omega_n) \to u^*(P_0) \text{ as } n \to \infty. \tag{3.35}$$

By applying now Equation (3.32) with ω_n, we can conclude that (3.34) holds.

It now follows that the ϕ_n's converge uniformly on compact subsets of $\mathcal{A}$ to

$$\Phi_{\mathcal{A}}(z) = \exp\big(\frac{2\pi}{\text{period}(u^*)}\big(u(z) + iu^*(z)\big)\big). \tag{3.36}$$

We end the proof by recalling a classical result (see for instance [20, Section 7] or [58, Theorem 4.3]) which asserts that Φ is a conformal homeomorphism between the interiors of $\mathcal{A}$ and $\mathcal{E}_{\mathcal{A}}$, respectively.

□

The case of continuous boundary. In this paragraph, we will briefly indicate why the boundary regularity assumption in Theorem 3.12 can be relaxed. Assume that $\mathcal{A}$ is a planar annulus, where $\partial\mathcal{A}$ is a union of disjoint, non-degenerate, continuous Jordan curves.

Definition 3.37 ([44, I.6.7]). We will say that a sequence of planar annuli $\mathcal{R}_j \subset \mathcal{R}$, $j = 1, 2, \ldots$, with $\{\Gamma_j^1, \Gamma_j^2\}$ as the components of their complements *converges from the inside* to an annulus $\mathcal{R}$ with $\{\mathcal{R}_1, \mathcal{R}_2\}$ as components of its complement, if for every $\epsilon > 0$ there exists n_ϵ such that for $n \geq n_\epsilon$ every point of $(\Gamma_j^i)_{i=1,2}$ lies within a spherical distance less than ϵ of the set $(\mathcal{R}_i, \mathcal{R}_2)_{i=1,2}$.

A classical construction due to Kellogg [41, Chapter XI.14] grants us an existence of a nested sequence of annuli, $\{\mathcal{A}_i\}$, where for all $i > 0$, $\{\mathcal{A}_i\} \subsetneq \mathcal{A}$, the boundary of $\mathcal{A}_i$ is polygonal, and the sequence converges to $\mathcal{A}$ from the inside. Furthermore, since each $\mathcal{A}_i$ is made of a lattice of squares, it is easy to construct a sequence of qausi-uniform triangulation each $\mathcal{A}_i$ which satisfy assumptions (V0)-(V3). Thus, $\mathcal{A}$ is presented as an increasing union of open subsets, the interiors of the $\mathcal{A}_i$, and each conformal embedding $\Phi_{\mathcal{A}_i}$ can be approximated according to Theorem 3.12. It follows that (up to normalization of the maps $\Phi_{\mathcal{A}_i}$), a subsequence of the $\{\Phi_{\mathcal{A}_i}\}$ will converge uniformly on compact subsets of $\mathcal{A}$, to its uniformizing map (see for instance [51, Lemma 2.2] or [8, Page 223]). Hence, we have the following

Corollary 3.38. *With the additional approximation processes described in the paragraph above, we may assume in Theorem 3.12 that $\partial\mathcal{A}$ is continuous.*

4. The simply connected case

In this section we will affirm Stephenson's conjecture (Conjecture 0.2) which was originally stated for the case of a bounded, simply connected, planar domain. Our point of departure is Theorem 3.12 whose notation will be closely followed. The proof of this case entails successively applying this theorem to an increasing sequence of annuli, a known modification of Koebe's compactness theorem, Riemann's removable singularity theorem, a lemma concerning the monotonicity of periods, and a basic covering property of planar Riemann surfaces.

In the following, we will let

$$\sigma(z) = \frac{1}{z}, \tag{4.1}$$

be the standard inversion of $\mathbb{C}$; it is well known that σ is conformal. We can now turn to

Theorem 4.2. *Let Ω be a simply connected domain, embedded in $\mathbb{C}$ and bounded by a closed, polygonal curve Γ; let $p_0 \in \Omega$ be a fixed point. Let $\{\Omega_n\} \subset \Omega$ be a nested sequence of disjoint, polygonal, Jordan disks with polygonal boundaries $\{\Theta_n\}$ such that the disks converge to p_0, that is,*

$$\Omega_1 \supset \Omega_2 \ldots \supset \Omega_k \ldots, \tag{4.3}$$

$$mesh(\Omega_n) \to 0 \text{ as } n \to \infty, \text{ and} \tag{4.4}$$

$$p_0 = \cap_n \Omega_n. \tag{4.5}$$

For each n, let $\mathcal{A}_n = \mathcal{A}_n(\Omega, \Theta_n)$ be the polygonal annulus defined by $\Omega \setminus \Omega_n$ with $\partial\mathcal{A}_n = \Gamma \cup \Theta_n$, endowed with a sequence of quasi-uniform triangulations $\{\mathcal{T}_{m,\mathcal{A}_n}\}_{m=1}^\infty$, such that for all $m = m(\mathcal{A}_n)$ large enough, $\mathcal{T}_{m,\mathcal{A}_n}$ satisfies the hypotheses of Theorem 3.12. Let

$$\Psi_n : \mathcal{A}_n \to \mathcal{E}_n \tag{4.6}$$

be the sequence of conformal homeomorphisms constructed according to Equation (3.15) onto the interior of concentric Euclidean annuli $\mathcal{E}_n$, whose inner and outer radii are given by, respectively

$$\{R_1, R_{2,n}\} = \{1, \exp\big(\frac{2\pi}{\text{period}(u_n^*)}\big)\}, \tag{4.7}$$

where u_n^* is the (smooth) harmonic conjugate of u_n, the solution of the boundary value problem (2.9) defined on $\mathcal{A}_n$.

Then, a normalized subsequence of $\{\sigma \circ \Psi_n\}$ converges uniformly on compact subsets of $\Omega \setminus p_0$ to a holomorphic homeomorphism Ψ from $\Omega \setminus p_0$ onto $\mathbb{D} \setminus 0$. Furthermore, Ψ can be extended to be holomorphic over Ω.

Proof. By construction, the $\{\mathcal{A}_n\}$ is a strictly increasing sequence, that is,

$$\mathcal{A}_1 \subsetneq \mathcal{A}_2 \subsetneq \ldots \subsetneq \mathcal{A}_k \ldots \tag{4.8}$$

which all share $\Gamma = \partial\Omega$ as their outer boundary component, and with $\Omega \setminus \{p_0\}$ being their union. The following lemma is needed in order to understand a monotonicity property of the sequence $\{A_n\}$.

Lemma 4.9. *The sequence $\{period(u_n^*)\}$ is strictly decreasing.*

Proof. By Green's theorem, for all $n > 1$ we have that,

$$\int_{\mathcal{A}_n} |\nabla u_n|^2 dx + \int_{\mathcal{A}_n} \Delta u_n u_n dx = \int_{\partial\mathcal{A}_n} \frac{\partial u_n}{\partial n} ds. \tag{4.10}$$

However, by the definition of period(u_n), and since u_n is the solution of the boundary value problem (2.9) defined on $\mathcal{A}_n$, for all $n > 1$, we have that

$$\int_{\mathcal{A}_n} |\nabla u_n|^2 dx = \text{period}(u_n). \tag{4.11}$$

It is clear that for all $n > 1$, u_n can be extended to be zero on $\mathcal{A}_{n+1} \setminus \mathcal{A}_{n+1}$ to a piecewise smooth function on $\mathcal{A}_{n+1}$ having the same boundary values as those of u_{n+1}. The assertion of the lemma now follows by the well-known characterization of u_{n+1} as the unique minimizer of the Dirichlet integral over $\mathcal{A}_{n+1}$.

$\boxed{\text{Lemma 4.9}}$

It follows from Equation (4.1), Equation (4.7) and the Lemma, that the sequence $\{A_n = \sigma(\mathcal{E}_n)\}$ consists of planar, concentric, Euclidean annuli, such that the inner and outer radii of each A_n are given by

$$\{r_1, r_{2,n}\} = \{1/\exp\big(\frac{2\pi}{\text{period}(u_n^*)}\big), 1\}, \tag{4.12}$$

respectively; where the sequence $\{r_{2,n}\}$ is strictly increasing. Note that all the A_n's share $\mathbb{S}^1 = \partial\mathbb{D}$ as their outer boundary component,

$$A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_k \ldots, \tag{4.13}$$

and the sequence $\{A_n\}$ exhausts $\mathbb{D} \setminus 0$.

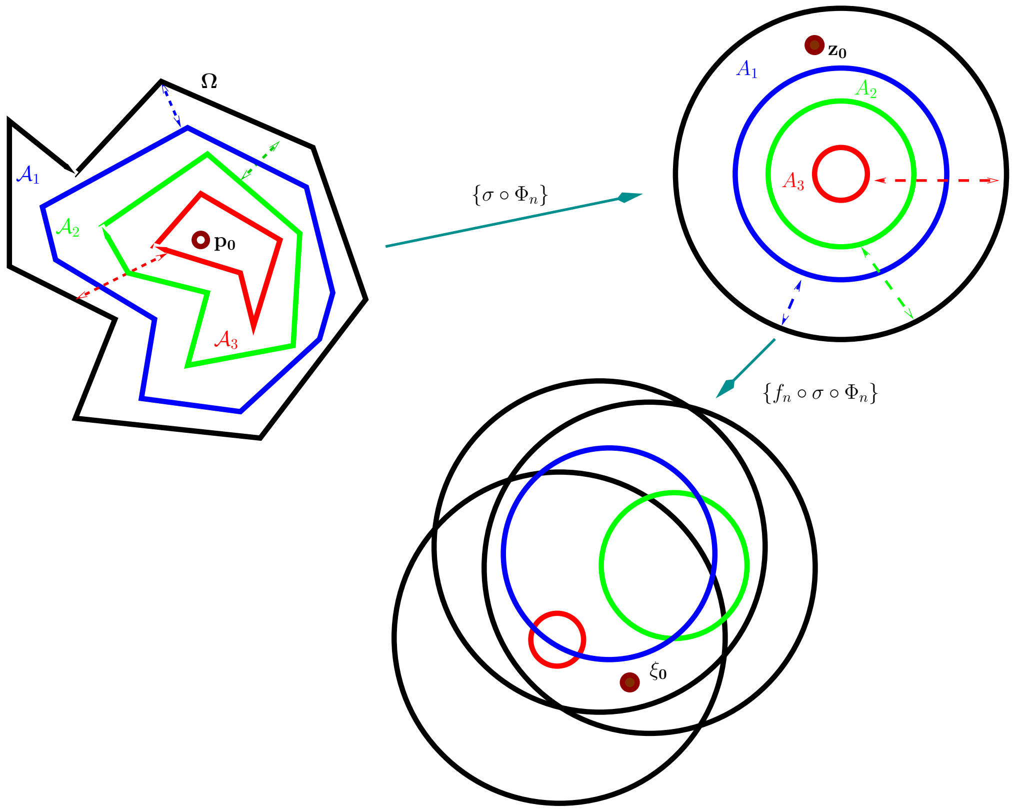

FIGURE 4.14. The evolution of Ω.

Pick $z_0 \in A_1$ and a local complex parameter at z_0. For all $n > 1$, we now apply a normalization by post composing $\sigma \circ \Psi_n$ with a conformal embedding $f_n : A_n \to \mathbb{C}$ so that the composed maps

$$\Upsilon_n = f_n \circ \sigma \circ \Psi_n : \mathcal{A}_n \to \mathbb{C} \tag{4.15}$$

satisfy

$$\Upsilon_n(z_0) = \xi_0 \text{ and } \Upsilon_n'(z_0) = 1. \tag{4.16}$$

Note that the image of each A_n is still a concentric Euclidean annulus, yet the sequence $\{\Upsilon_n(\mathcal{A}_n)\}$ is not (generically) concentric. Nevertheless, it follows from a modification of Koebe's compactness theorem (see for instance [19, Proposition 7.5]) and a Cantor diagonalization process, that a subsequence of the $\{\Upsilon_n\}$ converges uniformly on compact subsets of $\Omega \setminus p_0$, to a conformal, univalent mapping

$$\Upsilon : \Omega \setminus p_0 \to \mathbb{C}, \tag{4.17}$$

which is obviously not constant. It is also evident that Υ is bounded, and therefore, by Riemann's removable singularity mapping theorem, can be extended to a conformal, univalent, embedding from Ω. Hence, the extended map must be equal to the Riemann mapping with the same normalization. This ends the proof of the Theorem.

□

Remark 4.18. Following the rationale preceding Corollary 3.38, we can assume that Γ is continuous.

We conclude the main body of this paper with a few comments.

1. Disk packing and quasi-uniform triangulations. It is well known (see for instance [22, Section 5]) that a sequence of disk packings satisfying some minor conditions, induce (as explained in A.8) a sequence of quasi-uniform triangulations, that will in addition satisfy assumptions (V0) and (V1). However, assumption (V2) will not always be satisfied; it will be satisfied (for instance) for sub-packings of scaled copies of the infinite hexagonal disk packing (which were the subject of Thurston's original conjecture). Recall that assumption (V2) was used in the proof of Theorem 3.12 *only* in the part addressing the convergence of the $\bar{g}_n^*$. We will relax this assumption in [39].

2. The case of higher connectivity. As mentioned in the introduction, Stephenson's original conjecture can be formulated for any finitely connected, Jordan domain. However, several issues need to be addressed before an appropriate statement can be made. For instance, the existence of singular points and level curves for smooth harmonic functions solving a Dirichlet problem (analogous to the one in Theorem 3.12) on such domains needs to be addressed. We will treat this issue and others in [39].

Appendix A. preparatory facts

A.1. **The finite volume method.** In this section, we will assume that Ω is a fixed, bounded, m-connected $m \geq 2$), polygonal domain in $\mathbb{R}^2$. The vertex centered finite volume method (FVM) is a powerful discretization scheme, aimed at constructing and presenting approximations of solutions of partial differential equations in the form of algebraic set of equations, where the unknowns are placed at the vertices of a discrete grid. The phrase "finite volume" refers to *control volumes* associated with a chosen neighborhood of each vertex. This method turns out to be useful in the case of elliptic problems of diffusion type, such as

$$\nabla \cdot (K \nabla u) = f \text{ in } \Omega,\ f \in L^2(\Omega), \tag{A.1}$$

with prescribed boundary conditions on $\partial\Omega$. Here, the diffusion coefficient $K(x)$ is a 2×2 symmetric matrix with components in $L^\infty(\Omega)$, which is uniformly positive definite in Ω.

By integrating over Ω and employing *Gauss divergence theorem*, it follows that the equation above can be written as

$$-\int_{\partial\Omega} (K\nabla u) \cdot \hat{n}\ ds = \int_\Omega f\ dx, \tag{A.2}$$

where $\hat{n}$ denotes the unit outwards normal vector of $\partial\Omega$, ds denotes arc length along $\partial\Omega$, and dx is the (standard) area measure in the plane.

A common theme in finite volume methods is to approximate the solution of Equation (A.2) by replacing it with integrating Equation (A.1) over control volumes, while taking into account the boundary conditions. Let us now describe an important special case of a boundary value problem which will be essential to the applications of this paper. We start with

Definition A.3. A triangulation $\mathcal{T}$ of Ω is a set of triangles T_i, $i = 1, \dots, n$ such that the following hold.

$$\bar{\Omega} = \bigcup_{i=1}^{n} T_i, \text{ and } T_i \cap T_j = \emptyset, \text{ a vertex or one common edge, for all } i \neq j. \tag{A.4}$$

The following quantity is associated with a fixed triangulation.

Definition A.5. Let $\mathcal{T}$ be a triangulation on Ω, the mesh size of $\mathcal{T}$ is equal to

$$\sup_{T \in \mathcal{T}} d(T), \tag{A.6}$$

where $d(T)$ denotes the diameter of T (i.e., the length of its largest edge). Henceforth, $\mathcal{T}_\rho$ will denote a triangulation of Ω of mesh size equal to ρ.

In order to apply the machinery of numerical approximation of elliptic boundary value problems, one needs to avoid situations where triangles in (any) $\mathcal{T}_\rho$ become flat as $\rho \to 0$. To this end, we let $\sigma(T)$ denote the diameter of the largest circle that can be inscribed in a triangle T. We now define the geometric property of the class of triangulations that will be used throughout this paper.

Definition A.7. A family of triangulations $\{\mathcal{T}_\rho\}$ of Ω is called *quasi-uniform* (or *quasi-regular*) when $\rho \to 0$, if there exists a positive constant τ such that

$$\frac{d(T)}{\sigma(T)} \leq \tau, \text{ for all } T \in \mathcal{T}_\rho, \text{ and for all } \rho. \tag{A.8}$$

The induced *control volumes*, or the *Voronoi cells* used in this paper, which are associated with $\mathcal{T}$ are defined as follows. For each triangle $T \in \mathcal{T}$, let c_T denote the *circumcenter* of T, which by definition is the intersection point of the perpendicular bisectors of the edges. We join $c_{T'}$ to c_T by a segment $[c_{T'}, c_T]$ whenever T and T' share an edge. This procedure divides each (interior) triangle T into three quadrilaterals and induces a new decomposition of Ω. The control volume Ω_v of a vertex $v \in T$ is defined to be the star of v in this decomposition.

Henceforth in this paper, we will assume that

(V0): every quasi-uniform triangulation $\mathcal{T}$ under consideration is quasi-uniform and consists exclusively of *nonobtuse* triangles.

It then follows (see for instance [3, Theorem 6.5]) that $\mathcal{T}$ is a *Delaunay triangulation*, i.e., no point in the vertex set of $\mathcal{T}$ lies inside the circumcircle of any triangle in $\mathcal{T}$, and the corresponding Voronoi diagrams can be constructed by means of the perpendicular bisectors of the triangles' edges.

Let $\{\mathcal{T}_\rho\}_{\rho>0}$ be a family of triangulations of Ω, and let the set of vertices of $\mathcal{T}_\rho$ be enumerated by $\{v_1^\rho, v_2^\rho, \dots, v_{M(\rho)}^\rho\}$. Let $V_\rho(T)$ denote the set of vertices of $T \in \mathcal{T}_\rho$, and let $V_\rho^0(\mathcal{T}_\rho)$ denote the set of interior vertices of $V_\rho(\mathcal{T}_\rho) = \cup_{T \in \mathcal{T}_\rho} V_\rho(T)$. We now turn into integrating Equation (A.2) over each control volume $\Omega_{v_i}, i = 1, \dots, M(\rho)$. For each vertex v_i, $i = 1, \dots, M(\rho)$, let Λ_i denote the set of indices of neighboring vertices of v_i. Each Ω_{v_i} is an open, simply connected, and polygonally bounded set. Its boundary, $\partial\Omega_{v_i}$, consists of finitely many (straight) line segments $\Gamma_{i,j} = \partial\Omega_{v_i} \cap \partial\Omega_{v_i}$, $j = 1, \dots, n_i$, where n_i is the number of vertices adjacent to v_i, along which the normal $\hat{n}|\Gamma_{i,j} = \hat{n}_{i,j}$ is constant.

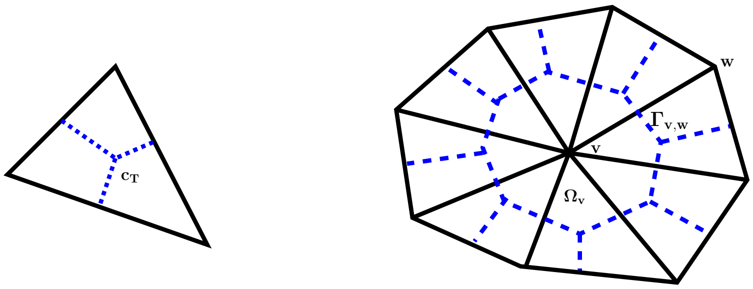

FIGURE A.9. A triangle and a control volume.

Definition A.10. Let $m_{i,j}$ denote the length of $\Gamma_{i,j}$, and let $d_{i,j} = |v_i - v_j|$, where $|,|$ denotes the Euclidean distance in $\mathbb{R}^2$.

Note that by the discussion preceding Definition A.11, the length of $\Gamma_{i,j}$ is equal to $|c_T - c_{T'}|$, where T and T' are the (only) two triangles that $\Gamma_{i,j}$ intersects. The union $\cup_v \Omega_v$ is called the *Voronoi* diagram of $\mathcal{T}_\rho$.

We now consider the homogeneous Dirichlet boundary condition defined in Ω by letting K be the identity matrix in Equation (A.1), and $\phi = 0$ as the boundary condition. Given a triangulation $\mathcal{T}$ of Ω, the finite dimensional space defined below is used.

Definition A.11.

$$\mathbb{V}_{0,\mathcal{T}} = \{v : \Omega \to \mathbb{R} \; | v \in C(\Omega), \; v|T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T} \text{ and } v = 0 \text{ on } \partial\Omega\}, \tag{A.12}$$

where $\mathcal{P}_1(T)$ denotes the space of linear polynomials in two variables over T.

One important feature of $\mathbb{V}_{0,\mathcal{T}}$ is that it is a linear subspace of a certain Sobolev space in which the *weak solution* of the above boundary value problem is constructed. Let us recall the following definition.

Definition A.13. The Sobolev space $H^{1,2}(\Omega)$ is the subset of $L^2(\Omega)$ defined by

$$H^{1,2}(\Omega) = \{v \in L^2(\Omega) \mid \partial_x v, \partial_y v \in L^2(\Omega)\}, \tag{A.14}$$

where $\partial_x v, \partial_y v$ denote the distributional derivatives of v in the x and the y directions, respectively. The integration is with respect to the standard Lebesgue measure in the plane which will be denoted by dx.

For $u, v \in H^{1,2}(\Omega)$, one defines the scalar product and an associated norm, respectively, by

$$(u, v)_{1,2} = \int_\Omega (uv + \nabla u \cdot \nabla v), \;\; |u|_{1,2}^2 = (u, u)_{1,2} \tag{A.15}$$

where $\nabla v = (\partial_x v, \partial_y v)$, and the scalar product is the Euclidean one in $\mathbb{R}^2$. It is well known that $H^{1,2}(\Omega)$ equipped with this scalar product is a Hilbert space. Finally, let $H_0^{1,2}(\Omega)$ be defined as the closure of $C_0^\infty(\Omega)$ in $H^{1,2}(\Omega)$ with respect to this norm. Equipped with this scalar product $H_0^{1,2}(\Omega)$ is a Hilbert space as well. It is a fact that $\mathbb{V}_{0,\mathcal{T}}$ is a linear subspace of $H_0^{1,2}(\Omega)$, whose dimension is equal to the cardinality of $\mathcal{T}^{(0)}$, the 0-skeleton of $\mathcal{T}$.

The first step in finite volume methods amounts to finding $u_\rho \in \mathbb{V}_{0,\mathcal{T}_\rho}$ such that

$$\int_{\partial\Omega_v} \nabla u_\rho \cdot \hat{n}\, ds = \int_{\Omega_v} f\, dx, \text{ for all } \ v \in V_\rho^0(\mathcal{T}_\rho), \tag{A.16}$$

where $\hat{n}$ denotes the outer unit normal to $\partial\Omega_v$. In order to ease the notation, we will suppress the index ρ when a fixed triangulation $\mathcal{T}_\rho$ is under consideration. Next, one studies the convergence in (various norms) of the u_ρ's to u, as the mesh of each triangulation converges to zero. This analysis is crucial to the applications of this paper and in the next section we will recall an approximation result from this theory.

A.2. **Piecewise linear approximations of the solution.** In this section, we will follow closely the exposition in [14, Sections 1-3] and apply a special type of *finite element method* in the case of homogenous Dirichlet boundary value problem, with the goal of writing it in a variational form.

Let

$$Y_\rho = \{\eta \in L_2(\Omega)) \mid \eta|\Omega_v \text{ is constant for } v \in V_\rho^0(\mathcal{T}_\rho), \text{ and } \eta|\Omega_v = 0 \text{ if } v \in \partial\Omega\}. \tag{A.17}$$

The following is the Petrov-Galerkin formulation of the finite volume method of Equation (A.2). Find $u_\rho \in \mathbb{V}_{0,\mathcal{T}_\rho}$ such that

$$\alpha_\rho(u_\rho, \eta) := -\sum_{v \in V_\rho^0(\mathcal{T}_\rho)} \eta(v) \int_{\partial\Omega_v} \nabla u_\rho \cdot \hat{n}\, ds = \int_\Omega f\eta\, dx, \quad \text{for all } \eta \in Y_\rho. \tag{A.18}$$

Let χ_v be the characteristic function of Ω_v, and let $I_\rho : C(\Omega) \to Y_\rho$ be the interpolation operator defined by

$$I_\rho \phi = \sum_{v \in V_\rho^0(\mathcal{T}_\rho)} \phi(v)\chi_v. \tag{A.19}$$

Recalling Definition A.11, we note that Equation (A.16) can be transformed to its variational form

$$\alpha_\rho(u_\rho, I_\rho\phi) = -\sum_{v \in V_\rho^0(\mathcal{T}_\rho)} I_\rho\phi(v) \int_{\partial\Omega_v} \nabla u_\rho \cdot \hat{n}\, ds = \sum_{v \in V_\rho^0(\mathcal{T}_\rho)} \phi(v) \int_{\Omega_v} f\, dx, \text{ for all } \phi \in \mathbb{V}_{0,\mathcal{T}_\rho}. \tag{A.20}$$

Note that for every $f \in L_p$ and $\phi \in \mathbb{V}_{0,\mathcal{T}_\rho}$, the following holds

$$\sum_{v \in V_\rho(\mathcal{T}_\rho)} \phi(v) \int_\Omega f\chi_v\, dx = \sum_{v \in V_\rho^0(\mathcal{T}_\rho)} \phi(v) \int_{\Omega_v} f\, dx. \tag{A.21}$$

Let us denote the right-hand side of Equation (A.21) by $(f, I_\rho\phi)$. Hence, Equation (A.20) can be written in the form

$$\alpha_\rho(u_\rho, I_\rho\phi) = (f, I_\rho\phi) \text{ for all } \phi \in \mathbb{V}_{0,\mathcal{T}_\rho}. \tag{A.22}$$

The existence of the solution u_ρ follows from the coerciveness of the inner-product α_ρ for ρ, small enough (see for instance [13]).

For the applications of this paper, it is necessary to consider *non-convex* polygonal domains. We will construct sequences of such domains with uniform upper and lower bounds on their largest and smallest angles, respectively, in order to approximate (in a sense that we will explain later) a given Jordan domain. In particular, due to the presence of corner singularities of vertex angles that are bigger than π, the recent L_∞ error analysis of the finite volume element which is needed for our applications is quite subtle (see for instance [14]).

We need to introduce some notation before stating the main result which we will be using. Let Ω be a bounded, (possibly) non-convex, polygonal domain. It is well known, that if $f \in L_p(\Omega)$, $1 < p < \infty$, then the solution u of the boundary value problem in (A.1) is *not* always in $H^2(\Omega)$ (see for instance [29, Section 2]). However, it turns out that u always belongs to a fractional order Sobolev space $H^{1+s}(\Omega)$ for some $0 < s < 1$; where s is effectively determined by the maximal interior angle of Ω and p (see [14, Section 2] for precise definitions).

The following foundational result that was obtained by Chatzipantelidis and Lazarov is the main numerical approximation result which will be used in this paper.

Theorem A.23 ([14, Theorem 4.8])**.** *Let u and u_ρ be the solutions of (A.1) and (A.16), respectively, with $f \in L_p, p > 1$. Then there exists a constant C, independent of ρ, such that*

$$\|u - u_\rho\|_{L_\infty} \leq C\rho^s \log(1/\rho)\|f\|_{L_p}. \tag{A.24}$$

A.3. **Stephenson's conductance constants - a finite volume method perspective.** In this section, we will express each u_ρ as the potential of an electrical network. To this end, two steps are necessary. First, we will present the form α_ρ in terms that relate to the geometry of the given triangulation $\mathcal{T}_\rho$. Second, we will turn the triangulation into a *finite network*. The second step amounts to assigning *conductance* constants along the edges of $\mathcal{T}_\rho$. The values of the conductance constants will be extracted from the first step.

Thus, we will see that u_ρ can be presented as the solution of a system of finitely many linear equations. Finally, from this presentation, we will define a quantity, the *discrete flux* of u_ρ, which will be important for the applications of this paper (see [36, 37] and [38, Section 1] for a background on finite networks and discrete flux in the above setting).

In [4], it was shown that under the assumption that K is the identity matrix the following holds

$$\alpha_\rho(\psi, I_\rho\phi) = \int_\Omega \nabla\psi \cdot \nabla\phi \, dx, \text{ for all } \psi, \phi \in \mathbb{V}_{0,\mathcal{T}_\rho}. \tag{A.25}$$

Hence, Equation (A.22) takes the form

$$\int_\Omega \nabla u_\rho \cdot \nabla\phi \, dx = (f, I_\rho\phi) \text{ for all } \phi \in \mathbb{V}_{0,\mathcal{T}_\rho}. \tag{A.26}$$

(It is well known that for every $\rho > 0$, $\mathbb{V}_{0,\mathcal{T}_\rho}$ is a finite dimensional vector space which is spanned by $\{\phi_i\}_\rho$ (the nodal basis). Therefore, it is sufficient to check the equation holds for any two elements in the nodal basis.)

We keep the notation as in the discussion preceding Figure A.9. The following Lemma and its corollary allow us to turn each $\mathcal{T}_\rho$ into a finite network.

Lemma A.27 ([3, Lemma 6.8]). *Let $\mathcal{T}_\rho$ be a triangulation of Ω, and consider its corresponding Voronoi diagram. Then, for an arbitrary triangle $T \in \mathcal{T}_\rho$ with vertices $v_i, v_j (i \neq j)$, the following relation holds*

$$\int_T \nabla\phi_i \cdot \nabla\phi_j \; dx = -\frac{m_{ij}^T}{d_{ij}}, \tag{A.28}$$

where m_{ij}^T is the length of the segment of Γ_{ij} that intersects T.

A computation then shows that

Corollary A.29 ([3, Corollary 6.9]). *Under the assumptions of Lemma A.27, we have*

$$\int_\Omega \nabla u_\rho \cdot \nabla\phi_i \; dx = \sum_{j\in\Omega_i} \frac{m_{ij}}{d_{ij}}\big(u_\rho(v_i) - u_\rho(v_j)\big). \tag{A.30}$$

Hence, by letting the index i range over the indices of the interior vertices (i.e, those that are in $V_\rho^0(\mathcal{T}_\rho)$), (A.26) turns into the following *system of linear equations*

$$\sum_{j\in\Omega_{v_i}} \frac{m_{ij}}{d_{ij}}\big(u_\rho(v_i) - u_\rho(v_j)\big) = \int_{\Omega_{v_i}} f \; dx. \tag{A.31}$$

Remark A.32. Note that when $f \equiv 0$, u_ρ is a *discrete* harmonic function on $\mathcal{T}_\rho^{(0)}$ with the conductance constant $\frac{m_{ij}}{d_{ij}}$ for the edge joining v_i to v_j.

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